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and the chord FE passes through P; and so for any circle. Q. E. D.

This problem was also solved by *G. B. M. Zerr*, *John B. Faught*, *J. F. W. Scheffer*, and *O. W. Anthony*.

PROBLEMS.

37. Proposed by B. F. BURLESON, Oneida Castle, New York.

Inscribe in a semicircle a rectangle having a given area; a rectangle having the maximum area.

38. Proposed by LEONARD E. DICKSON, M. A., Fellow in Mathematics, University of Chicago.

Give a *strictly geometric* proof of my fundamental theorem on the Inscription of Regular Polygons, viz: Suppose a circle of unit radius divided at the points $A, A_1, A_2, A_3, \dots A_p, \dots$ into $2p+1$ equal parts and the diameter AO drawn. Then, if the chords $OA_1, OA_2, \dots OA_p$ be drawn, we have $OA_1-OA_2+OA_3-OA_4+OA_5-\dots\pm OA_p=1$.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

25. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

A leaf of the curve: "The Devil on Two Sticks", equation $y^4-x^4+100a^2x^2-96a^2y^2=0$, revolves around the axis of x. Deduce the expression for the volume generated.

I. Solution by the PROPOSER.

From the equation of the given curve, we deduce $y^2=48a^2\pm\sqrt{(2304a^4-100a^2x^3+x^4)}$....(1); that is, $(PD)^2=48a^2+\sqrt{(2304a^4-100a^2x^2+x^4)}$, and, therefore, $(P'D)^2=48a^2-\sqrt{(2304a^4-100a^2x^2+x^4)}$. Hence the expression for the volume generated after the curve has made a complete revolution around the axis of x, becomes

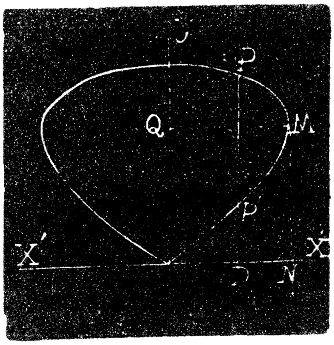
$$V=2\pi\left[\int_0^{6a}[48a^2+\sqrt{(2304a^4-100a^2x^2+x^4)}]dx-\right.$$

$$\left.\int_0^{6a}[48a^2-\sqrt{(2304a^4-100a^2x^2+x^4)}]dx\right]\dots.(2).$$

Condensing (2), then factoring, etc.,

$$V=4\pi\int_0^{6a}\surd(2304a^4-100a^2x^2+x^4)dx=192\pi a^2$$

$$\int_0^{6a}\sqrt{\left[\left(1-\frac{x^2}{36a^2}\right)\left(1-\frac{x^2}{64a^2}\right)\right]}dx\ldots.(3).$$



I. Let $x^2/36a^2=w^2$; then will $x^2/64a^2=\frac{9}{16}w^2,=c^2w^2$, and $dx=6adw$. Making these substitutions in (3), we have

$$V=1152\pi a^3\int_0^1\surd[(1-w^2)(1-c^2w^2)]dw$$

$$=1152\pi a^3\int_0^1\frac{1-(1+c^2)w^2+c^2w^4}{\surd[(1-w^2)(1-c^2w^2)]}dw$$

$$=384\pi a^3\int_0^1\left[\frac{1-2(1+c^2)w^2+3c^2w^4}{\surd[(1-w^2)(1-c^2w^2)]}\right.$$

$$\left.+\frac{2-(1+c^2)w^2}{\surd[(1-w^2)(1-c^2w^2)]}\right]dw$$

$$=384\pi a^3\left[\int_0^1\frac{1-2(1+c^2)w^2+3c^2w^4}{\surd[(1-w^2)(1-c^2w^2)]}dw\right.$$

$$\left.+\left(\frac{1+c^2}{c^2}\right)\int_0^1\frac{\surd(1-c^2w^2)}{\surd(1-w^2)}dw-\left(\frac{1-c^2}{c^2}\right)\int_0^1\frac{dw}{\surd[(1-w^2)(1-c^2w^2)]}\right]$$

$$=384\pi a^3\left\{\left[w\surd(1-w^2)(1-c^2w^2)\right]_0^1+\left(\frac{1+c^2}{c^2}\right)\left[\mathrm{E}(c,w)_0^1\right.\right.$$

$$\left.\left.-\left(\frac{1-c^2}{1+c^2}\right)\mathrm{F}(c,w)_0^1\right]\right\}\ldots.(4).$$

II. By making $x^2/36a^2=\sin^2\psi$, $x^2/64a^2=\frac{9}{16}\sin^2\psi,=c^2\sin^2\psi$, and $dx=6a\cos\psi d\psi$, we easily deduce from (3) the following expression:

$$V=1152\pi a^3\int_0^{\frac{1}{2}\pi}\cos^2\psi\surd(1-c^2\sin^2\psi)d\psi=384\pi a^3\left(\frac{1+c^2}{c^2}\right)\left[\mathrm{E}(c,\tfrac{1}{2}\pi)\right.$$

$$\left.-\left(\frac{1-c^2}{1+c^2}\right)\mathrm{F}(c,\tfrac{1}{2}\pi)\right]\ldots.(5),$$ which is identical with the right-hand member of (4). As a *working-result*, the right-hand member of (5) is preferable to the right hand member of (4). Expanding the Legendrian elliptic-integrals in the right hand member of (5), uniting corresponding terms, etc., we have

$$V=288\pi^2a^3[1-\frac{1}{8}c^2-\frac{1}{64}c^4-\text{etc.}]=\frac{136359}{512}\pi^2a^3\ldots.(6).$$ Putting $a=1$ and remembering that $\pi^2=9.8696+$, we obtain $V=2628.533+$.

II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

The polar equation to the curve is,

$$r^2=\frac{4a^2(24\sin^2\theta-25\cos^2\theta)}{\sin^4\theta-\cos^4\theta}=\frac{4a^2(24\sin^2\theta-25\cos^2\theta)}{\sin^2\theta-\cos^2\theta}=\frac{2a^2(1+49\cos 2\theta)}{\cos 2\theta}.$$

Both leaves are equal and each is symmetrical with reference to the y axis. Also, the area of the upper leaf is comprised between the limits $\theta=\frac{\pi}{4}$ and $\theta=\frac{3\pi}{4}$. Let A=area of this leaf, $\bar{y}$=ordinate of its centroid. Then volume required is $V=2\pi\bar{y}A$.

$$A=\tfrac{1}{2}\int r^2\,d\theta=a^2\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{1+49\cos 2\theta}{\cos 2\theta}\,d\theta=\frac{49\pi a^2}{2}.$$

$$\bar{y}=\tfrac{2}{3}\frac{\int r^3\sin\theta\,d\theta}{\int r^2\,d\theta}=\frac{2a\sqrt{2}}{3}\,\frac{\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left\{\frac{1+49\cos 2\theta}{\cos 2\theta}\right\}^{\frac{3}{2}}\sin\theta\,d\theta}{\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{1+49\cos 2\theta}{\cos 2\theta}\,d\theta}.$$

$$\bar{y}=\frac{4a\sqrt{2}}{147\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left\{\frac{1+49\cos 2\theta}{\cos 2\theta}\right\}^{\frac{3}{2}}\sin\theta\,d\theta.$$

$\therefore\quad V=\dfrac{4\pi a^3\sqrt{2}}{3}\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left\{\frac{1+49\cos 2\theta}{\cos 2\theta}\right\}^{\frac{3}{2}}\sin\theta\,d\theta.$ Let $\sqrt{2}\cos\theta=\cos\phi$;

$$\therefore\quad V=\frac{4\pi a^3}{3}\int_0^{\pi}(48-49\cos^2\phi)^{\frac{3}{2}}\operatorname{cosec}^2\phi\,d\phi=196\pi a^3\int_0^{\pi}(48-49\cos^2\phi)^{\frac{1}{2}}\cos^2\phi\,d\phi$$

$=784\sqrt{3}\pi a^3\displaystyle\int_0^{\pi}(1-c\cos^2\phi)^{\frac{1}{2}}\cos^2\phi\,d\phi$, where $c=\dfrac{49}{48}$,

$$=784\sqrt{3}\pi a^3\int_0^{\pi}(1-\tfrac{1}{2}c\cos^2\phi-\tfrac{1}{8}c^2\cos^4\phi-\tfrac{1}{16}c^3\cos^6\phi-\tfrac{5}{128}c^4\cos^8\phi-\text{,etc.})\cos^2\phi\,d\phi$$

$$=392\sqrt{3}\pi^2a^3\left\{1-\frac{3}{8}\left(\frac{49}{48}\right)-\frac{5}{64}\left(\frac{49}{48}\right)^2-\frac{35}{1024}\left(\frac{49}{48}\right)^3-\frac{315}{16384}\left(\frac{49}{48}\right)^4-\text{, etc.}\right\}$$

$=392\sqrt{3}\pi^2a^3\left\{1-3b-5b^2-\dfrac{35}{2}b^3-\dfrac{315}{4}b^4-\ldots\ldots\right\}$, where $b=\dfrac{49}{384}$.

The fifth term of this serie $=\frac{1}{50}$ nearly. Or thus: but since $c=e^2$ is not less than unity, let $x=\cos\phi$, then

$$\int_0^{\pi}(1-c\cos^2\phi)^{\frac{1}{2}}\cos^2\phi\,d\phi=2\int_0^{1}\left\{\frac{e^2x^2-1}{x^2-1}\right\}^{\frac{1}{2}}x^2\,dx=2I.$$

Let $S=\sqrt{(e^2x^2-1)(x^2-1)}$, then $d(Sx)=\left(S+\dfrac{2e^2x^4-x^2(e^2+1)}{S}\right)dx$

$$=\frac{1+3e^2x^4-2x^2(e^2+1)}{S}\,dx=\frac{1-(2e^2-1)x^2}{S}\,dx+3dI=\frac{1-\frac{2e^2-1}{e^2}}{S}\,dx$$

$$-\frac{2e^2-1}{e^2}d[H(e,x)]+3dI=-\frac{e^2-1}{e^2}d[H'(e,x)]-\frac{2e^2-1}{e^2}d[H(e,x)]+3dI.$$

$$\therefore\left[x\sqrt{(e^2x^2-1)(x^2-1)}+\frac{e^2-1}{e^2}H'(e,x)+\frac{2e^2-1}{e^2}H(e,x)\right]_0^1=3I.$$

$$2I=\left[\frac{2}{3}\cdot\frac{e^2-1}{e^2}H'(e,x)+\frac{2}{3}\cdot\frac{2e^2-1}{e^2}H(e,x)\right]_0^1.$$

$$\therefore\ V=\frac{1568\sqrt{3}\pi a^3}{3e^2}\left[\frac{2e^2-1}{e^2}H(e,x)+\frac{e^2-1}{e^2}H'(e,x)\right]_0^1,\text{ but } e^2=\frac{49}{48},$$

$\therefore\ V=\dfrac{512\sqrt{3}\pi a^3}{49}\left[50H\left(\dfrac{7}{4\sqrt{3}}\right)+H'\left(\dfrac{7}{4\sqrt{3}}\right)\right]$, where H and H' denote the hyperbolic functions corresponding to the elliptic functions E and F.

III. Remarks by Professor J. F. W. SCHEFFER. A. M., Hagerstown, Maryland.

This curve, called in French "la courbe du diable," is of the middle point of a chord to the equilateral hyperbola $x^2-y^2=2a^2$, the chord being of constant length and equal to seven times the transverse axis $2a\sqrt{2}$. Its equation is found thus: Let (x_1, y_1) and (x_2, y_2) be the extremities of a chord, and (x, y) any point of the curve, then we have the equations: $x^2-y^2=2a^2$....(1). $x^2_2-y^2_2=2a^2$....(2), $2x=x_1+x_2$....(3), $2y=y_1+y_2$....(4), and $(x_1-x_2)^2+(y_1-y_2)^2=392a^2$....(5). Subtracting (2) from (1), and considering (3) and (4), we have $(x_1-x_2)x=(y_1-y_2)y$, whence $y_1-y_2=(x_1-x_2)\dfrac{x}{y}$. Substituting this in (5), we get $x_1-x_2=\dfrac{14\sqrt{2}.ay}{(x^2+y^2)^{\frac{1}{2}}}$, $y_1-y_2=\dfrac{14\sqrt{2}.ax}{(x^2-y^2)^{\frac{1}{2}}}$, and combining these with (2) and (3), we get $x_1=x+\dfrac{7\sqrt{2}.ay}{(x^2+y^2)^{\frac{1}{2}}}$, $y_1=y+\dfrac{7\sqrt{2}.ax}{(x^2+y^2)^{\frac{1}{2}}}$. Substituting in (1) and simplifying we finally have $y^4-x^4-96a^2y^2+100a^2x^2=0$.

Query: Can any one furnish a reason for the peculiar name of the "devil's curve," or the name which Prof. Matz employes?

Also solved by *Prof. C. W. M. Black.*

PROBLEMS.

34. Proposed by GEORGE LILLEY, Ph. D., LL. D., Park School, Portland, Oregon.

A hare is at O, and a hound at E, 40 rods east of O. They start at the same instant each running with uniform velocity. The hare runs north. The hound runs directly towards the hare and overtakes it at N, 320 rods from O. How far did the hound run?

35. Proposed by H. C. WHITAKER, B. S., C. E., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Water is running into a vessel in the shape of a frustum of a cone (radii up-